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EFFECTIVE DIFFUSION OF A DYNAMICALLY PASSIVE IMPURITY IN NARROW CHANNELS

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The diffusion equation is rarely solved successfully by analytical means when it contains a convective term in which the velocity components are complex functions of the space coordinates. In the case of diffusion in channels, the author of [1] proposed a method of reducing the basic equation to a simpler form containing an effective diffusion (dispersion) coefficient. This approach was later followed intensively (see [2-4], for example, where other approaches to the problem were also proposed). Here, we obtain a similar equation of effective diffusion in narrow channels under the condition that the stream function in the channel used to express the dispersion factor is known. Calculation of the stream function is an independent problem. We subsequently use the relations obtained to solve the problem of extracting a substance from narrow trenches (slits) when the channel has a boundary through which exchange of the substance with the main flow is possible.

As is known, the flow scheme of Lavrent'ev [5] agrees better with experimental results than does other models for the flow of a low-viscosity fluid in a trench. The flow model is based on the theorem [6, 7] of constancy of vorticity in closed regions. However, vorticity may not be constant when the viscosity coefficient μ is variable [8]. Assuming that the vorticity distribution was known, we obtained a general expression for the stream function in a narrow cavity bounded by the coordinate lines of an orthogonal coordinate system. As an example, we examined the case of extraction of a substance from a deep slit.

We propose an integral transformation which can be used to solve a certain range of problems of the dispersion of a substance in channels.

1. Derivation of Equation of Effective Diffusion and Initial Condition. We will assume that the length of the channel in the X_1 direction is much greater than the length in the X_2 direction. The boundaries of the channel are assumed to coincide with the coordinate lines of the plane X_1, X_2. We will limit ourselves to the two-dimensional problem. Let the stream function Ψ in the channel be known, and let its values at the boundaries of the channel be equal to zero. Then the components of the velocity of the fluid in the channel are determined by the formulas

$$v_1 = H_2^{-1}\partial\Psi/\partial X_2,\ v_2 = -H_1^{-1}\partial\Psi/\partial X_1, \tag{1.1}$$

where $H_{1,2}(X_1, X_2)$ are the Lamé constants. The equation of diffusion of the impurity in the channel has the form

$$\varepsilon^2 H_1 H_2 \frac{\partial c}{\partial t} + \varepsilon W(\psi, c) = \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2}\frac{\partial c}{\partial x_2}\right) + \varepsilon^2 n \frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial c}{\partial x_1}\right),$$
$$W(\psi, c) = \frac{\partial\psi}{\partial x_2}\frac{\partial c}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial c}{\partial x_2}, \tag{1.2}$$

while the dimensionless parameters and coordinates are connected to the dimensional parameters and coordinates by the relations

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$$x_1=\frac{X_1}{l_1},\quad x_2=\frac{X_2}{l_2},\quad \psi=\frac{\Psi}{\Psi_0},\quad t=\frac{\tau\Psi_0^2}{Dl_1^2},\quad \varepsilon=\frac{\Psi_0 l_2}{Dl_1},\quad n=\frac{D^2}{\Psi_0^2}.$$

Here, c is the concentration of the substance; l_1 and l_2 are the dimensions of the channel in the directions of the X_1 and X_2 axes; τ is dimensional time; Ψ_0 is the characteristic scale of the stream function; D is the coefficient of molecular diffusion. If we assume that the quantity n is on the order of unity or greater, then the "narrowness" condition of the channel will be expressed by the inequality $l_1 \gg l_2$ or $\varepsilon \ll 1$. The choice of time scale was suggested to us by Taylor's successive approximation procedure [1] for the analogous problem in a channel when the nonsteady term is ignored in the first approximation and considered in the second approximation. It can be shown that in the case of flow in the channel, the procedure proposed below leads to the Taylor dispersion equation in the first approximation if we start out on the basis of the corresponding diffusion equation analogous to (1.2). We assume that the walls of the channel are impermeable to the impurity, i.e., that we have the boundary conditions

$$\partial c/\partial x_2 |_{x_2=0;1} = 0, \tag{1.3}$$

while at the initial moment of time we have

$$c|_{t=0} = g(x_1, x_2). \tag{1.4}$$

We will not concretize the condition on the boundary $x_1 = 0$ linking the channel with the external flow or the condition on the bottom of the channel $x_1 = 1$. It is natural to solve problem (1.2)-(1.4) by the method of small perturbations [9, 10] in the form of a series

$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \ldots, \tag{1.5}$$

Having inserted this series into Eq. (1.2) and having grouped terms of the same order with respect to ε, we obtain

$$\partial[(H_1/H_2)\partial c_0/\partial x_2]/\partial x_2 = 0; \tag{1.6}$$

$$\partial[(H_1/H_2)\partial c_1/\partial x_2]/\partial x_2 = W(\psi, c_0); \tag{1.7}$$

$$\frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2}\frac{\partial c_i}{\partial x_2}\right) = W(\psi, c_{i-1}) + H_1H_2\frac{\partial c_{i-2}}{\partial t} - n\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial c_{i-2}}{\partial x_1}\right),\quad i = 2, 3 \tag{1.8}$$

We note that Eqs. (1.6)-(1.8) do not contain derivatives of the sought functions with respect to time (at $i \geq 2$, the equations contain the time derivatives of functions which must be determined earlier by successive integration of equations with smaller numbers). This makes the problem a singularly perturbed problem [9, 10] and requires the construction of certain characteristics of the internal solution.

The integral of Eq. (1.6) satisfying condition (1.3) has the form $c_0 = F(x_1, t)$. Then (1.7) is simplified, and its first integral will be

$$\partial c_1/\partial x_2 = (\psi H_2/H_1)\partial F/\partial x_1, \tag{1.9}$$

where we used the triviality of the stream function at the walls $x_2 = 0;\ 1$. Now we integrate Eq. (1.8) for x_2 within (0, 1) at $i = 2$. After certain calculations, we obtain the equation

$$m(x_1)\partial F/\partial t = \partial\{[D_1(x_1) + D_2(x_1)]\partial F/\partial x_1\}/\partial x_1 \tag{1.10}$$

to determine the function $F = c_0$. Here

$$m(x_1) = \int_0^1 H_1H_2\,dx_2,\quad D_1(x_1) = \int_0^1 \psi^2\frac{H_2}{H_1}\,dx_2,\quad D_2(x_1) = n\int_0^1 \frac{H_2}{H_1}\,dx_2. \tag{1.11}$$

For simplicity, we assume that everywhere in the region $x_1, x_2 \in [0; 1], 0 < \delta \leqslant H_1, H_2 \leqslant N < \infty$, so that all of the integrals (1.11) converge. Similarly, we find the following from (1.9) and (1.8) at $i = 3$

$$c_1 = \frac{\partial F}{\partial x_1}\int_0^{x_2}\psi\frac{H_2}{H_1}\,dx_2 + G(x_1, t), \tag{1.12}$$

where the function G is determined by the equation

$$m(x_1)\frac{\partial G}{\partial t}=\frac{\partial}{\partial x_1}\left[(D_1+D_2)\frac{\partial G}{\partial x_1}\right]+\frac{\partial}{\partial x_1}\left[D_3(x_1)\frac{\partial^2 F}{\partial x_1^2}\right]+$$

$$+n\frac{\partial^2 F}{\partial x_1^2}\int\limits_0^1\frac{H_2}{H_1}\lambda\,dx_2+n\frac{\partial F}{\partial x_1}\int\limits_0^1\frac{H_2}{H_1}\frac{\partial\lambda}{\partial x_1}dx_2-\frac{\partial^2 F}{\partial x_1\,\partial t}\int\limits_0^1 H_1H_2\lambda\,dx_2, \tag{1.13}$$

$$\lambda(x_1,x_2)=\int\limits_0^{x_2}\psi\frac{H_2}{H_1}dx_2,\ \eta(x_1,x_2)=\int\limits_0^{x_2}\frac{H_2}{H_1}\psi^2dx_2,\ D_3(x_1)=\int\limits_0^1\left(\lambda\frac{\partial\eta}{\partial x_2}-\eta\frac{\partial\lambda}{\partial x_2}\right)dx_2,$$

i.e., by (1.10), but with a nonvanishing source term expressed through the function F.

If we cannot describe the boundary of the channel by the coordinate line of a certain orthogonal coordinate system but we can satisfy the inequality $l'_{\pm}(x_1) \ll 1$, where $x_2 = l_{\pm}(x_1)$ are relations determining the boundaries of the channel and the coordinates x_1 and x_2 are Cartesian coordinates in this case, then instead of (1.10) we can use the same method to obtain

$$l(x_1)\partial F/\partial t=\partial[l(x_1)D_4(x_1)\partial F/\partial x_1]/\partial x_1,$$

where the function $D_4(x_1)$ is determined by the formula

$$D_4(x_1)=n+\int\limits_{l_-}^{l_+}\psi^2dx_2,\quad l=l_++l_-.$$

We find the initial condition for (1.10) by adding to the internal solution, for which we change the time $T = t/\varepsilon^2$. The equation of zeroth order with respect to ε appears as follows in the internal variables

$$H_1H_2\partial C_0/\partial T=\partial[(H_1/H_2)\partial C_0/\partial x_2]/\partial x_2 \tag{1.14}$$

(C is the internal concentration). Having integrated Eq. (1.14) over x_2 with (0, 1) and using additional conditions (1.3), (1.4), we find

$$\frac{\partial}{\partial T}\int\limits_0^1 H_1H_2C_0dx_2=0,\ \text{т. е.}\ \int\limits_0^1 H_1H_2C_0dx_2=\int\limits_0^1 H_1H_2g(x_1,x_2)\,dx_2. \tag{1.15}$$

Using the principle of limiting combination [9]

$$\lim_{T\to\infty}C_0=\lim_{t\to 0}c_0=F|_{t=0} \tag{1.16}$$

in Eq. (1.15), we obtain the sought initial condition

$$F|_{t=0}=\frac{1}{m(x_1)}\int\limits_0^1 H_1H_2g(x_1,x_2)\,dx_2=\langle g\rangle. \tag{1.17}$$

We find the initial condition for (1.13) as follows. We subject the equation $H_1H_2\partial C_1/\partial T + W[\psi, C_0] = \partial(H_1/H_2)\cdot\partial C_1/\partial x_2]/\partial x_2$ to a Laplace transformation with respect to the variable T for the function of the first approximation of ε of the internal solution, and we integrate the resulting relation over x_2 within (0, 1). With allowance for boundary conditions (1.3) and the conditions $\psi|_{x_2=0;\,1} = 0$

$$p\int\limits_0^1 H_1H_2C_1^*dx_2=\frac{d}{\partial x_1}\int\limits_0^1\psi\frac{\partial C_0^*}{\partial x_2}dx_2, \tag{1.18}$$

where the asterisk denotes quantities subjected to Laplace transformation; p is the Laplace transform variable. Henceforth using the limit equation [11]

$$\lim_{p\to 0}pf^*=\lim_{T\to\infty}f, \tag{1.19}$$

we only use information for the function $\partial C_0^*/\partial x_2$ which is asymptotic at $p \to 0$. This information is obtained from the Laplace transform of Eq. (1.14). Combination with the external solution and condition (1.17) shows that the dominant term of the asymptotic expansion of C_0^* at $p \to 0$ will be $p^{-1}\langle g\rangle$. However, this term does not depend on x_2, so it does not contribute to the right side of (1.18). The following term R is on the order of unity with respect to p. For it, we have the expression

$$\frac{\partial R}{\partial x_2} = \frac{H_2}{H_1} \int_0^{x_2} H_1 H_2 [\langle g \rangle - g] \, dx_2. \tag{1.20}$$

Subsequent terms in the asymptotic expansion of C_0^* at $p \to 0$ will be of a lower order with respect to p, so their contribution at the limit of integral (1.18) will be zero for $p \to 0$. Expansion of the external solution in the internal variables $c(T\varepsilon^2) = c_0|_{t=0} + \varepsilon c_1|_{t=0} + \varepsilon^2 [c_2|_{t=0} + T\partial c_0/\partial t|_{t=0}] + \ldots$ shows that the principle of limiting combination (1.16) can also be used for the functions c_1 and C_1. Changing over to the limit $p \to 0$ in (1.18) and using (1.19) and Eqs. (1.12) and (1.20), we find the sought initial condition for the function G in (1.13):

$$G|_{t=0} = \frac{1}{m(x_1)} \frac{d}{dx_1} \left\{ \int_0^1 \lambda H_1 H_2 [g - \langle g \rangle] \, dx_2 - \langle g \rangle \int_0^1 H_1 H_2 \lambda \, dx_2 \right\}. \tag{1.21}$$

Thus, we have formulated Eqs. (1.10) and (1.13) and the corresponding initial conditions for the functions c_0 and c_1 of the first two approximations. It should be noted that the above procedure is easily generalized to the case of the presence of sources of the substance both within the channel and on its walls if the sources are of low intensity (on the order of ε^2).

2. Calculation of the Stream Function and the Effective Diffusion Coefficient. The equation for the stream function in the channel will be

$$\partial[(H_1/H_2)\partial\psi/\partial x_2]/\partial x_2 + \varepsilon^2 n \partial[(H_2/H_1)\partial\psi/\partial x_1]/\partial x_1 = -H_1 H_2 \omega \tag{2.1}$$

with the boundary conditions

$$\psi|_{x_2=0;1} = 0, \ \psi|_{x_1=0;1} = 0 \tag{2.2}$$

($\omega = \Omega(X_1, X_2) l_2^2/\Psi_0$ is dimensionless vorticity). By having ε in (2.1) approach zero, we obtain the equation of the external problem. We write the solution of this equation, satisfying the first condition of (2.2), in the form

$$\psi_+(x_1, x_2) = \left[\int_0^{x_2} \frac{H_2}{H_1} dx_2 \Big/ \int_0^1 \frac{H_2}{H_1} dx_2 \right] \int_0^1 \frac{H_2}{H_1} dx_2 \int_0^{x_2} H_1 H_2 \omega(x_1, \xi) \, d\xi - \int_0^{x_2} \frac{H_2}{H_1} d\lambda \int_0^{\lambda} H_1 H_2 \omega(x_1, \xi) \, d\xi, \tag{2.3}$$

where the "+" and "−" subscripts denote the external and internal solutions, respectively. The internal equation is obtained from (2.1) by changing over to the internal variable $x_3 = x_1/\varepsilon n^{1/2}$ and then passing to the limit $\varepsilon \to 0$:

$$\frac{\partial}{\partial x_2} \left[r(x_2) \frac{\partial \psi_-}{\partial x_2} \right] + \frac{1}{r(x_2)} \frac{\partial^2 \psi_-}{\partial x_3^2} = -H_1 H_2 \omega|_{x_1=0} \tag{2.4}$$

$[r(x_2) = H_1/H_2|_{x_1=0}]$. We introduce a new variable x_4 and a new function ω_1 through the formulas

$$\omega_1(x_4) = H_1^2 \omega|_{x_1=0}, \ x_4 = \int_0^{x_2} dx/r(x). \tag{2.5}$$

The function $x_4(x_2)$ has an inverse due to monoticity, since $r > 0$. We also take advantage of this, having expressed ω_1 through x_4. Now we write the equation for ψ_- as

$$\partial^2\psi_-/\partial x_4^2 + \partial^2\psi_-/\partial x_3^2 = -\omega_1(x_4), \ x_4 \in (0, \alpha), \ x_3 \in (0, \infty) \tag{2.6}$$

$[\alpha = x_4(1) = D_2(0)/n]$, while the boundary conditions

$$\psi_-|_{x_4=0;\alpha} = 0, \ \psi_-|_{x_3=0} = 0, \ \psi_-|_{x_3\to\infty} \to \psi_+[x_2(x_4), 0] = \psi_*(x_4). \tag{2.7}$$

It should be noted that the function $\psi_*(x_4)$ satisfies Eq. (2.6), so it is easily reduced to a homogeneous equation. Its solution, obtained by the Fourier method, has the form

$$\psi_- = \psi_*(x_4) - \frac{2\alpha}{\pi^2} \sum_{k=1}^{\infty} \exp\left(-\frac{\pi k x_3}{\alpha}\right) \frac{\sin\left(\frac{\pi k x_4}{\alpha}\right)}{k^2} \int_0^{\alpha} \omega_1(\xi) \sin\left(\frac{\pi k \xi}{\alpha}\right) d\xi. \tag{2.8}$$

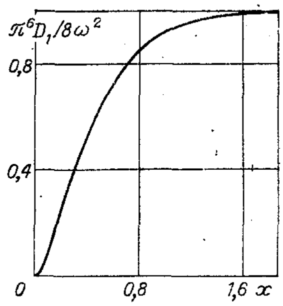

Fig. 1

The general part of the internal and external solutions is equal to $\psi_*(x_4)$, so that the following is a uniformly valid zeroth-order expansion in ε for the function ψ:

$$\psi = \psi_+(x_1, x_2) - \frac{2\alpha}{\pi^2}\sum_{k=1}^{\infty}\frac{\exp\left(-\frac{\pi k x_3}{\alpha}\right)}{k^2}\sin\left(\frac{\pi k x_4}{\alpha}\right)\int_0^{\alpha}\omega_1(\xi)\sin\left(\frac{\pi k \xi}{\alpha}\right)d\xi. \tag{2.9}$$

It should be noted that we can similarly construct the internal solution near the boundary $x_1 = 1$. Here, Eq. (2.9) is augmented by yet another sum. This sum coincides with the sum already obtained, to within the accuracy of the notation.

We will use Eq. (2.9) to calculate the coefficient D_1 for the case of a deep slit in Cartesian coordinates. We will also assume that ω = const. Then $H_1 = H_2 = 1$, $x_4 = x_2$, $\alpha = 1$, and after insertion of the function $\psi_+ = \omega x_2(1 - x_2)/2$ into (2.9) and the resulting function ψ into (1.11), we obtain the following for the coefficient D_1:

$$D_1 = \frac{8\omega^2}{\pi^6}\sum_{k=0}^{\infty}\{1 - \exp[-(2k+1)\pi x_3]\}^2/(2k+1)^6. \tag{2.10}$$

Series (2.10) converges very rapidly and, in practice, a single term of the sum can be used for calculations. The graph of the function D_1 is shown in Fig. 1.

One of the main applications of the formulas obtained here is study of diffusion processes in channels with a flow in accord with the scheme in [5]. In this case, the flow velocity v is usually assigned far from the channel, and ω is the sought variable. Various algorithms have been proposed to calculate ω [12-14]. However, by taking advantage of the narrowness of the cavity, we can propose an approximate formula linking v and ω if we equate v to the velocity at the point $x_2 = 0.5$, $x_3 = 0$. We find the following expression from (2.9) (v directed against the x_2 axis)

$$v = |\partial\psi/\partial x_3|\,|_{x_3=0;x_2=0,5} = 0.37\omega \tag{2.11}$$

for the dimensionless velocity vector.

3. Example. We will examine the equalization of concentrations in a channel in an external flow, having assumed that the concentration of the impurity on the boundary of the slit is equal to zero. This assumption is valid for sufficiently high velocities of the main flow. For the sake of brevity, we designate the Cartesian coordinates in the usual manner: x and y. The case $n \ll 1$ is typical at high velocities, and we will limit ourselves to this case. As the calculations and the flow patterns in channels [5, 12] show, the streamline bounding the regions of vortex and potential flow in accord with the scheme in [5] is fairly close to the line $y = 0$. Thus, for simplicity, we take the streamline to be a straight line $y = 0$. The same conclusion can be reached for narrow channels in the general case as well.

We write the equation of effective diffusion in the form

$$\partial c/\partial t = \partial\{[n + D_1(y)]\partial c/\partial y\}/\partial y. \tag{3.1}$$

We take the initial concentration of impurity in the slit to be unity, so that the additional conditions for Eq. (3.1) will be

$$c|_{t=0} = 1,\ c|_{y=0} = 0,\ \partial c/\partial y|_{y=1} = 0. \tag{3.2}$$

By virtue of the smallness of n, it is natural to seek the solution of problem (3.1), (3.2) by the perturbation method. With conditions (3.2), the solution of the external problem (far from the boundaries y = 0 and 1) and the internal problem near the line $y = 1$ is constant and is equal to unity. Our main interest is in the local solution near the line $y = 0$, for which the coefficient D_1 in (3.1) is determined by Eq. (2.10). It should be noted that the function $D_1(y)$ has a second-order zero at $y = 0$. This suggests the form of the internal variable: $x = y(k/n)^{1/2}$ [$k = D_1''(0)/2 = \omega^2/6$]. The equation of the internal problem takes the form

$$\partial c/\partial \zeta = \partial[(1 + x^2)\partial c/\partial x]/\partial x \tag{3.3}$$

with the additional conditions

$$c_{\zeta=0} = 1,\ c_{x=0} = 0,\ c_{x\to\infty}\ -\ \text{fin.},\ \zeta = kt.$$

Use of the Laplace transform with respect to the variable ζ, while keeping the notation employed in Part 1, leads to the problem

$$d[(x^2 + 1)dc^*/dx]/dx = pc^* - 1; \tag{3.4}$$

$$c^*|_{x=0} = 0,\ c^*,\ dc^*/dx|_{x\to\infty}\ -\ \text{fin.} \tag{3.5}$$

with the solution

$$c^*(p, x) = 1/p - Q_\nu(ix)/pQ_\nu(+0i),\ \nu = -0.5 + \sqrt{0.25 + p} \tag{3.6}$$

[$Q_\nu(x)$ is a spherical second-order function]. The singular points of the function $c^*(p, x)$ are the pole $p = 0$ and the branch point $p = -0.25$. The Reimann–Mellin integral

$$c(\zeta, x) = \frac{1}{2\pi i}\int_L c^* \exp(p\zeta)\,dp$$

(integration is done over the straight line $\operatorname{Re} p = \beta > 0$), by virtue of the asymptotic formula in [15] for the function $Q_\nu(x)$ and the Jordan lemma, can be reduced to the residue at the point $p = 0$ and to the integrals over the edges of the slit, connecting the points $p = \infty$ and -0.25 along the negative part of the real axis of the plane p. A similar slit in the plane x $(-\infty, 1)$ was used to isolate the single-valued branch of the function $Q_\nu(x)$, so that the symbol $Q_\nu(+0i)$ should be taken as the value of the function $Q_\nu(x)$ on the upper edge of the slit. Using well-known [15] formulas for $Q_\nu(+0i)$, the formulas linking first- and second-order Legendre functions, and the residue of the function $\exp(p\zeta)c^*(p, x)$ at the point $p = 0$, we have

$$c = \frac{2\operatorname{arctg} x}{\pi} - 4\sqrt{\pi}\exp\left\{-\frac{\zeta}{4}\right\}\int_0^\infty \frac{r\,\operatorname{th}(\pi r)\exp(-\zeta r^2)\,G_1(x, r)\,dr}{\operatorname{ch}(\pi r)\,(r^2 + 1/4)\,|\Gamma(1/4 + ir/2)|^2}, \tag{3.7}$$

where $\Gamma(x)$ is a gamma function, while the function $G_j(x, r)$ is determined by the formulas

$$G_j(x, r) = \frac{P_{ir-1/2}(ix) + (-1)^j P_{ir-1/2}(-ix)}{-2(-i)^j},\quad j = 1, 2, \tag{3.8}$$

$P_\nu(x)$ is a first-order Legendre function. We are interested mainly in the flow of the substance from the channel, which is found from the expression

$$q = \left.\frac{\partial c}{\partial x}\right|_{x=0} = \frac{2}{\pi} + 8\pi\exp\left\{-\frac{\zeta}{4}\right\}\int_0^\infty \frac{r\,\operatorname{th}(\pi r)\exp(-\zeta r^2)\,dr}{\operatorname{ch}(\pi r)\,(r^2 + 1/4)\,|\Gamma(1/4 + ir/2)|^4}. \tag{3.9}$$

In differentiating the integral (3.7), we changed the order of differentiation and integration. This was justified due to the absolute convergence of the integral and its derivatives with respect to x, which exists at $\zeta \geq \delta > 0$. We also used the equality $P_\nu'(0) = -2\sqrt{\pi}/\Gamma(-\nu/2)\Gamma(1/2 + \nu/2)$. The integral in (3.9) converges rapidly for large values of time. Using the Laplace method [11], we easily obtain a formula, asymptotic at $\zeta \to \infty$, for the dimensionless flow of the substance

$$q \sim \frac{2}{\pi} + \frac{8\pi^{5/2}}{\Gamma^4(1/4)}\frac{\exp(-\zeta/4)}{\zeta\sqrt{\zeta}} = 0.6366 + 0.8099\frac{\exp(-\zeta/4)}{\zeta\sqrt{\zeta}}, \tag{3.10}$$

showing the rate at which q approaches the steady-state value $2/\pi$. It can also be seen that $q \sim 1/(\pi\zeta)^{1/2}$ at $\zeta \to 0$.

The solution (3.7) is a uniformly valid solution of Eq. (3.1) of zeroth order with respect to n. It is of a boundary-layer character and describes the initial stage of recovery of the substance from the slit. The "depletion" of the channel can still be ignored, i.e., the external solution can be taken equal to unity. It is interesting to examine the behavior of the mean concentration of the substance in the channel for long times. To do this, we integrate (3.1) over y within (0, 1) and we change over to the dimensionless coordinates:

$$d\langle c\rangle/d\tau = -q\Psi_0/l_1^2,\ \langle c\rangle = \frac{1}{l_1}\int_0^{l_1} c\,dx_1. \tag{3.11}$$

It is evident from this that the characteristic time of change in the mean concentration $\langle c\rangle$ will be $T_2 = l_1^2/\Psi_0$. At the same time, the characteristic time scale for Eq. (3.3) and its solution (3.7) $T_1 = Dl_1^2/\Psi_0^2$. We additionally introduce the "diffusion" time $T_3 = l_1^2/D$. The following relation exists between this time scale and the chosen inequality $n \ll 1$

$$T_2/T_3 = (T_1/T_3)^{1/2} = D/\Psi_0 = n^{1/2} \ll 1,$$

which establish a hierarchy of scales: $T_1 \ll T_2 \ll T_3$. We see from this that, first of all, solution (3.7) is valid for sufficiently short times ($\tau \ll T_2$), when the change in concentration away from the boundary can still be ignored. Second, the presence of circulatory flow in the channel significantly shortens the characteristic time of change in $\langle c\rangle$, which in the absence of flow is on the order of T_3. Equations (3.7) and (3.9) show that the solution of the problem changes significantly only in the narrow strip $x_1 = l_1 O(n^{1/2}) \ll l_1$, i.e., $c = 1$ nearly throughout the channel. If we took another constant in place of unity in initial condition (3.2), then this constant would be present as a multiplier in Eq. (3.9) and the other formulas. We will designate this multiplier as c_∞. By virtue of the above remarks, the mean concentration coincides with c_∞ with a high degree of accuracy. Since the mean concentration changes very slowly in the scale T_1, then replacement of c_∞ by $\langle c\rangle$ in Eqs. (3.7) and (3.9) leads to a solution whereby Eq. (3.3) is satisfied almost exactly. The evolution of $\langle c\rangle$ can be determined with the same accuracy, having inserted the value of q, expressed through $\langle c\rangle$ in accordance with the last remark, into (3.11). We thus arrive at the equations

$$d\langle c\rangle/d\tau = -\varkappa\langle c\rangle,\quad \varkappa = 2\Psi_0/\pi l_1^2. \tag{3.12}$$

Changing over in (3.11) to an asymptotic expression for the flow of the substance is valid for times on the order of T_2, for which Eq. (3.12) applicable. Here, $\varkappa$ can be regarded as the coefficient of exchange between the main flow and the stagnant channel.

With the initial condition $\langle c\rangle|_{\tau=0} = 1$, Eq. (3.12) is easily integrated: $\langle c\rangle = \exp(-\varkappa\tau)$. We also write the expression for c

$$c_* = \langle c\rangle\, c, \tag{3.13}$$

which is valid throughout the range of times. Here, c is determined by Eq. (3.7). At $\tau \gg T_1$, Eq. (3.13) takes the simple form

$$c_* = (2/\pi)\exp(-\varkappa\tau)\,\mathrm{arctg}\left(x_1\omega/l_1\sqrt{6n}\right). \tag{3.14}$$

4. Some Additional Remarks and Generalizations. Since $\mathrm{Pe} = vl_1/D \gg 1$ for sufficiently high velocities of the external flow, then, because $\psi \sim vl_1$, $D_1 \sim v^2l_1^2/D \gg D$ or $n \ll 1$. Thus, Eq. (3.3) is typical for a channel of general form near the region of union of the vortex and potential flows. Then it makes sense to determine the coefficient $k = D_1''(0)/2$, as was shown in Sec. 3 for the general case as well. The sequence of calculation is as follows: we expand the function $\psi_*(x_4)$ in (2.8) into a Fourier series; then we arrive at the variable x_4 in Eq. (1.11) for D_1; using the Parseval equality, we obtain an expression for $D_1(x_1)$ in the form of a series; differentiating this expression twice and substituting $x_3 = 0$, we find a sum which, by expanding Green's function for the given problem, can be reduced to the form

$$k = \frac{D_1''(0)}{2} = \frac{4}{\alpha^2}\int_1^{\alpha}(\alpha - x)\,\omega_1(x)\,dx\int_0^{x} y\,\omega_1(y)\,dy. \tag{4.1}$$

We will mention one other useful formula for k. Since $x_1 = 0$, $\psi = 0$, and $v_2 \neq 0$ on the junction line, then we find from (1.1) that $\psi \sim -H_1 v_2|_{x_1=0}\, x_1$. Inserting this relation into the formula for $D_1(x_1)$, we find the sought formula

$$k=\int_0^1 H_1H_2v_2^2|_{x_1=0}\,dx_2=m(0)\,\langle v_2^2\rangle|_{x_1=0}. \tag{4.2}$$

We note that to find the parameters of the problem in the limiting case, it is sufficient to know the distribution of v_2 and ω_1 along the interface $x_1 = 0$ rather than over the entire flow region. This is a significant simplification. If we substitute the velocity of the external flow v into Eq. (4.2) and put ω = const in (4.1), then after we equate these formulas we obtain (Cartesian coordinates x_1, x_2) the relation $v \approx 0.41\omega$, linking v and ω, which is sufficiently close to (2.11).

Boundary conditions of the first, second, or third type can be assigned on the boundary x = 0 for the fundamental equation (3.3) in the junction region. An effective method of solving the problems created here, besides the Laplace transform, is the use of a special integral transform containing the function $G_j(x, r)$ (3.8). Using the method in [16], after certain transformations we obtain

$$f(x)=2\int_0^\infty \frac{r\,\mathrm{th}(\pi r)\,\Phi(r,x)\,dr}{\{[h^2+B^2(r)]\,\mathrm{ch}(\pi r)-2hB(r)\}}\int_0^\infty f(\xi)\,\Phi(r,\xi)\,d\xi, \tag{4.3}$$

the sought integral transform, where $h \geq 0$ is a constant parameter in the third-order boundary condition $\partial\Phi/\partial x|_{x=0} = h\Phi$,

$$B(r)=\frac{2\,\mathrm{tg}\left[\frac{\pi}{2}\left(ir-\frac{1}{2}\right)\right]\Gamma^2\left(\frac{3}{4}+\frac{ir}{2}\right)}{\Gamma^2\left(\frac{1}{4}+\frac{ir}{2}\right)},\quad \Phi(r,x)=B(r)\,G_2(x,r)+hG_1(x,r).$$

It is easily shown that B(r) and, thus, Φ, are real on the integration line. In special cases, $h \to \infty$ and 0, which corresponds to a transition from a third-order boundary condition to first- and second-order conditions; (4.3) becomes the well-known [17] expansions

$$f(x)=2\int_0^\infty \frac{r\,\mathrm{th}(\pi r)}{\mathrm{ch}(\pi r)}G_j(x,r)\,dr\int_0^\infty f(\xi)\,G_j(\xi,r)\,d\xi,\quad j=1,2.$$

It should be noted that the scale of the stream function Ψ_0 which has been frequently used here [such as in (3.12)] can be expressed through other parameters from dimensional relations for k [(4.1) or (4.2)].

In examining the example in Sec. 3, for simplicity we used a flow scheme in which ω is constant over the entire slit. In experiments, some zones with a different (constant) vorticity are seen (see [5], which presented a photograph of a flow with two zones and noted that vorticity is equal magnitude in the zones but opposite in sign).

The formulas used in Sec. 2 for the stream function to calculate the effective diffusion coefficient D_1 contain an arbitrary vorticity distribution in the slit. In particular, the distribution is piecewise-constant. This consideration can easily be taken into account in calculations if the boundaries of the zones and the vorticities in them are known. Since the local solution in the region where the vortex and potential flows join together is of decisive importance in the above-examined flow parameters, the foregoing consideration is unimportant for obtaining the fundamental boundary-layer equation in this region (3.3) and analyzing its solution. However, in the transition to large values of time (see Sec. 3) when several vortex zones are present, it is best to replace Eq. (3.12) by a system of N (according to the number of zones) equations. These equations are derived on the basis of the same considerations as (3.12) and appear as follows:

$$\begin{gathered}
d\langle c_1\rangle/d\tau=-\varkappa_0\langle c_1\rangle+\varkappa_1(\langle c_2\rangle-\langle c_1\rangle),\\
d\langle c_j\rangle/d\tau=\varkappa_{j-1}(\langle c_{j-1}\rangle-\langle c_j\rangle)+\varkappa_j(\langle c_{j+1}\rangle-\langle c_j\rangle),\ j=2,3,\ldots,N-1,\\
d\langle c_N\rangle/d\tau=\varkappa_N(\langle c_{N-1}\rangle-\langle c_N\rangle),
\end{gathered} \tag{4.4}$$

where we assume that the impurity concentration in the main flow is zero, and we ignore the possible exchange of the substance between the bottom of the channel and the last N-th zone. Averaging of c_j is done similarly to (3.11) over the entire j-th zone. System (4.4) is augmented by the natural initial conditions

$$\langle c_j\rangle|_{\tau=0}=c_j^0,\quad j=1,2,\ldots,N \tag{4.5}$$

and is easily solved by standard methods. The physical interpretation of problem (4.4), (4.5) is as follows. The assigned distribution of the impurity concentration over the channel first is quickly equalized (in the corresponding scale, see Sec. 3) within the circulation zones due to effective diffusion. This leads to conditions (4.5) as well. The subsequent evolution of the system is determined by the relatively slight transport of the substance over the boundaries of the zones, since the effective diffusion coefficient decreases sharply here and approaches the molecular diffusion coefficient. This exchange of substance between zones is also described by system (4.4).

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